Numerical-Perturbation Method for the Nonlinear Analysis of Structural Vibrations

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A numerical-perturbation method is proposed for the determination of the nonlinear forced response of structural elements. Purely analytical techniques are capable of determining the response of structural elements having simple geometries and simple variations in thickness and properties, but they are not applicable to elements with complicated structure and boundaries. Numerical techniques are effective in determining the linear response of complicated structures, but they are not optimal for determining the nonlinear response of even simple elements when modal interactions take place due to the complicated nature of the response. Therefore, the optimum is a combined numerical and perturbation technique. The present technique is applied to beams with varying cross sections.

I. Introduction

ANY large-amplitude deflection of a beam or a plate which is restrained at its ends or along its edges results in some midplane stretching. One must account for this stretching with nonlinear strain-displacement relationships. The nonlinear equations of motion describing this situation were the basis of a number of earlier investigations and are the basis for the present paper as well. The purpose of the present paper is to present a new scheme for determining the response to a harmonic excitation. Emphasis is placed on the case when the frequency of the excitation is near a natural frequency.

A convenient way to attack this nonlinear problem involves representing the deflection curve or surface with an expansion in terms of the linear, free-oscillation modes. The deflection is then determined in two steps. First, the damping, the forcing, and the nonlinear terms are deleted and the linear modes (eigenfunctions) and natural frequencies (eigenvalues) are determined. Second, the time-dependent coefficients in the expansion are obtained from a set of coupled, nonlinear, ordinary, second-order differential equations, the linear modes being used to determine the coefficients in these equations. (The procedure is described in detail in Sec. II.)

Generally, one cannot obtain the linear modes analytically for structural elements having complicated boundaries and composition, and one cannot easily determine the character of the time-dependent coefficients through numerical integration of the set of nonlinear equations. (The results obtained in the present numerical example are typical of the complicated manner in which the steady-state amplitudes of the various modes making up the response can vary with the amplitude and the frequency of the excitation.) Consequently, an optimal procedure involves a numerical method to determine the linear, free-oscillation modes and an analytical method to determine the time-dependent coefficients. The present procedure combines either a finite-element or a finite-difference method with the method of multiple scales (see, for example, Ref. 1). The following brief review mentions representative examples of the work that was and is being done, and it provides a background for the present scheme.

Using only one mode in the expansion of the deflection, Woinowsky-Krieger,[2] Burgreen,[3] Evensen,[4] Wrenn and Mayers,[5] Erigen,[6] Srinivasan,[7] and Aravamudan and Murthy[8] considered the free oscillations of beams with various boundary conditions. Eisley[9] and Chu and Herrmann[10] considered plates as well as beams, and Eisley[9] and Bauer[11] considered the forced responses. Generally, a Ritz-Galerkin procedure was used to determine a nonlinear equation describing the temporal variation of the amplitude, and this equation was solved exactly in terms of elliptic functions or approximately by the Lindstet-Poincaré technique. None of these studies considered modal interactions, though Chu and Herrmann did note the possibility.

MacDonald[12] apparently was the first to consider modal interactions. He assumed a multimode expansion and then found the coefficients in terms of elliptic functions. Using a somewhat different set of equations, Atluri[13] accounted for modal couplings in considering the effects of rotary and longitudinal inertia on the free oscillations of simply supported beams having ends which are free to move longitudinally. He found the time-dependent coefficients using the method of multiple scales. Earlier, Wagner,[14] using the Lindstet-Poincaré technique, considered similar problems, and Nayfeh,[15] using the method of multiple scales, considered the effect of axially varying properties on the free oscillations of beams with midplane stretching. Neither Wagner nor Nayfeh considered modal couplings. Dowell[16] and Morino[17] considered modal couplings in their nonlinear analyses of panel flutter. Dowell obtained the coefficients numerically, and Morino used the method of multiple scales.

Bennett and Eisley[18] used a multimode expansion in considering the forced response of a clamped-clamped beam to a harmonic excitation, and later Bennett[19] considered the ultraharmonic response of a simply supported beam. Tseng and Dugundji[20] also used a multimode expansion in considering the forced response of a clamped beam about its buckled configuration. They also considered ultraharmonic, subharmonic, and ultrasubharmonic responses. In these studies, the coefficients were determined by the method of harmonic balance, and apparently none considered the possibility of internal resonance. This phenomenon occurs when the natural frequencies are commensurable. In considering the forced response of a clamped-simply supported beam, Nayfeh, Mook, and Sridhar[21] illustrated the importance of considering internal resonance. The first and second natural frequencies for these beams are nearly in the ratio 1:3, and as a consequence, there is a strong interaction between the first two modes, which combine to make up the response in the first approximation.

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Some of the results in the abovementioned papers were compared, and found to be in close agreement, with the experimental data of Ray and Bert,[22] who considered the forced response of simply supported beams. There is no internal resonance and no strong modal interaction for such beams.

There are four salient features of the present procedure. First, the deflection curve or surface is represented with a multimodal expansion in terms of the linear, free-oscillation modes. Second, the linear modes and the natural frequencies are obtained numerically. Third, the temporal problem is solved by the method of multiple scales. Fourth, internal resonances are considered.

II. Description of the Procedure

A. General Comments

The equations describing the lateral vibrations of structural elements may be combined into an equation of the form

$$L(w)+\partial^2 w/\partial t^2 = \varepsilon[N(w)+D(w)+F(\mathbf{r},t)] \tag{1}$$

where w is the deflection, ε is a small parameter that appears when the equation is written in terms of dimensionless variables, L is a linear spatial operator, N is a nonlinear spatial operator, $D(w)$ is the damping term, and $F(\mathbf{r}, t)$ is the forcing term. The significance of considering the damping and the forcing terms to be of order ε is discussed below.

The deflection is subject to boundary conditions of the form

$$B(w) = 0 \quad \text{on} \quad \Gamma \quad \text{for all} \quad t \tag{2}$$

where B is a linear spatial operator which is independent of ε and Γ is the boundary of the structure. No initial conditions need to be specified because interest is restricted to the steady-state response.

The deflection is represented by an expansion in terms of the linear, free-oscillation modes

$$w(\mathbf{r},t) = \sum_n \psi_n(t)\phi_n(\mathbf{r}) \tag{3}$$

where the summation may be either one or two dimensional. The ϕ_n are solutions of

$$L(\phi_n) - \omega_n^2\phi_n = 0 \tag{4a}$$

and, from Eq. (2),

$$B(\phi_n) = 0 \quad \text{on} \quad \Gamma \tag{4b}$$

The equations for the ψ_n are obtained by substituting Eq. (3) into Eq. (1), multiplying the result by ϕ_m and the weighting function, and integrating over the region occupied by the structure. After using Eq. (4a), one can obtain a set of equations of the form

$$d^2\psi_m/dt^2 + \omega_m^2\psi_m = \varepsilon\left[\sum_n\sum_p\sum_q \alpha_{mnpq}\psi_n\psi_p\psi_q - c_m(d\psi_m/dt) + 2P_m\cos\lambda t\right] \tag{5}$$

With regard to Eq. (5), we recognize the nonlinearity due to midplane stretching is cubic and put

$$\int_R f(\mathbf{r})\phi_m N\left(\sum_n \phi_n\psi_n\right)dR = \sum_n\sum_p\sum_q \alpha_{mnpq}\psi_n\psi_p\psi_q$$

we assume the damping is modal and put

$$\int_R f(\mathbf{r})\phi_m D\left(\sum_n \phi_n\psi_n\right)dR = -2c_m\, d\psi_m/dt$$

and we consider the forcing function to be harmonic and put

$$\int_R f(\mathbf{r})\phi_m F(\mathbf{r},t)\,dR = 2P_m\cos\lambda t$$

Here, f is the weighting function and R is the region.

With the present procedure, the ϕ_n and ω_n are obtained either by solving Eqs. (4) with a finite-difference method or a finite-element method. Then the coefficients α_{mnpq}, c_m, and P_m are obtained by numerical quadrature, and finally an approximate solution valid for small, but finite, amplitudes of the ψ_n is obtained. Determining this approximate solution is discussed next.

A straightforward expansion of the form

$$\psi_m(t;\varepsilon) \sim \sum_{n=0} \varepsilon^n\psi_{mn}(t) \tag{6}$$

is not uniformly valid for large values of t because secular terms of the form $t^n\exp(i\omega_m t)$ appear in ψ_{mn}. It is necessary to modify the straightforward procedure in order to obtain a uniformly valid expansion. One such modification is the method of multiple scales.

According to this method, new independent variables (time scales) are introduced as follows:

$$T_n = \varepsilon^n t \tag{7}$$

The T_o is a fast scale associated with changes occurring with a frequency near ω_m, while the T_n for $n \geqq 1$ are slow scales associated with changes that can only be noticed after several cycles. Replacing the single time scale by multiple time scales results in the ordinary derivatives being transformed into an expansion in terms of partial derivatives as follows:

$$d/dt = D_o + \varepsilon D_1 + \cdots \tag{8a}$$

and

$$d^2/dt^2 = D_o^2 + \varepsilon 2D_oD_1 + \cdots \tag{8b}$$

where $D_n = \partial/\partial T_n$. Moreover, the expansion given in Eq. (6) is modified as follows:

$$\psi_m(t;\varepsilon) \sim \sum_{n=0} \varepsilon^n\psi_{mn}(T_o, T_1, T_2, \ldots) \tag{9}$$

Substituting Eqs. (8) and (9) into Eq. (5) and equating coefficients of the same powers of ε, we obtain a set of equations for each power of ε as follows:

for ε^o,

$$D_o^2\psi_{mo} + \omega_m^2\psi_{mo} = 0 \tag{10}$$

for ε^1,

$$D_o^2\psi_{m1} + \omega_m^2\psi_{m1} = \sum_n\sum_p\sum_q \alpha_{mnpq}\psi_{no}\psi_{po}\psi_{qo} - 2D_oD_1\psi_{mo} - 2c_mD_o\psi_{mo} + 2P_m\cos\lambda T_o \tag{11}$$

etc.

The solution to Eq. (10) can be written in the form

$$\psi_{mo} = A_m(T_1, T_2, \ldots)\exp(i\omega_m T_o) + cc \tag{12}$$

where cc represents the complex conjugate of the preceding term(s). At this stage, the A_m are unknown. They are to be determined at the next level of approximation by satisfying the so-called solvability condition.

Substituting Eq. (12) into Eq. (11) leads to

$$\begin{aligned} D^2\psi_{m1} + \omega_m^2\psi_{m1} = {} & P_m\exp(i\lambda T_o) - i\omega_m(2D_1A_m + 2c_mA_m) \times \\ & \exp(i\omega_m T_o) + \sum_n\sum_p\sum_q \alpha_{mnpq}\{A_nA_pA_q \times \\ & \exp[i(\omega_n+\omega_p+\omega_q)T_o] + A_n\bar{A}_p\bar{A}_q \times \\ & \exp[i(\omega_n-\omega_p-\omega_q)T_o] + A_n\bar{A}_pA_q \times \\ & \exp[i(\omega_n-\omega_p+\omega_q)T_o] + A_nA_p\bar{A}_q \times \\ & \exp[i(\omega_n+\omega_p-\omega_q)T_o]\} + cc \end{aligned} \tag{13}$$

The solvability condition requires the coefficient of $\exp(i\omega_m T_o)$ to vanish and hence leads to a set of coupled, first-order differential equations for the A_m. In this set, the equation associated with ψ_{m1} contains P_m if λ is near ω_m as well as the nonlinear coefficients of any terms for which

$$\omega_m \approx \omega_n + \omega_p + \omega_q \tag{14a}$$

$$\omega_m \approx \omega_n - \omega_p - \omega_q \tag{14b}$$

$$\omega_m \approx \omega_n - \omega_p + \omega_q \tag{14c}$$

The last combination when $p = q$, which always occurs when the nonlinearity is cubic, was considered by MacDonald and Atluri, while Nayfeh et al.[21] considered all possible combinations. These conditions are associated with the so-called internal resonance, except when $p = q$ in Eq. (14c).

As an example, we take $\lambda \approx \omega_2$ and $\omega_2 \approx 3\omega_1$. To describe these approximations quantitatively we introduce detuning parameters σ_1 and σ_2 defined as follows:

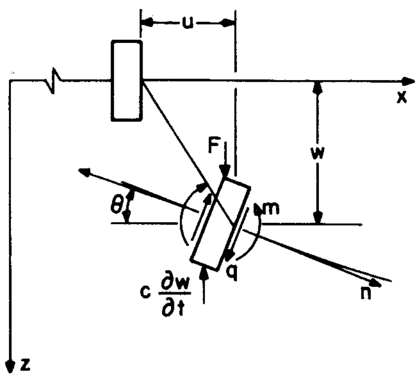

Fig. 1 Beam geometry and nomenclature.

$$\omega_2 = 3\omega_1(1+\varepsilon\sigma_1) \tag{15a}$$

and

$$\lambda = \omega_2(1+\varepsilon\sigma_2) \tag{15b}$$

For convenience, we write A_m in polar form

$$A_m = \tfrac{1}{2}a_m \exp(i\beta_m) \tag{15c}$$

where both a_m and β_m are real functions of T_1.

Putting Eqs. (15) into Eq. (13) and separating the result into real and imaginary parts, one finds the following solvability conditions:

$$\omega_1(a_1' + c_1 a_1) - \tfrac{1}{8}\delta_1 a_1^2 a_2 \sin\mu_1 = 0 \tag{16a}$$

and

$$\omega_1 a_1 \beta_1' + \tfrac{1}{8}\left(a_1 \sum_{j=1} \gamma_{1j} a_j^2 + \delta_1 a_1^2 a_2 \cos\mu_1\right) = 0 \tag{16b}$$

$$\omega_2(a_2' + c_2 a_2) + \tfrac{1}{8}\delta_2 a_1^3 \sin\mu_1 - P_2 \sin\mu_2 = 0 \tag{17a}$$

and

$$\omega_2 a_2 \beta_2' + \tfrac{1}{8}\left(a_2 \sum_{j=1} \gamma_{2j} a_j^2 + \delta_2 a_1^3 \cos\mu_1\right) + P_2 \cos\mu_2 = 0 \tag{17b}$$

and

$$a_m' + c_m a_m = 0 \tag{18a}$$

and

$$\omega_m a_m \beta_m' + a_m \sum_{j=1} \gamma_{mj} a_j^2 = 0 \tag{18b}$$

for $m \geqq 3$, where

$$\mu_1 = 3\omega_1\sigma_1 T_1 - 3\beta_1 + \beta_2 \tag{19a}$$

$$\mu_2 = \omega_2\sigma_2 T_1 - \beta_2 \tag{19b}$$

$$\delta_1 = \alpha_{1112} + \alpha_{1211} + \alpha_{1121} \tag{19c}$$

$$\delta_2 = \alpha_{2111} \tag{19d}$$

$$\gamma_{mj} = 2(\alpha_{mjmj} + \alpha_{mmjj} + \alpha_{mjjm}), \quad m \neq j \tag{19e}$$

$$\gamma_{mm} = 3\alpha_{mmmm} \tag{19f}$$

and primes denote differentiation with respect to T_1. In the present paper, we are only concerned with the first term in a uniformly valid expansion; hence all T_n for $n \geqq 2$ are considered constants.

For the steady-state response, all the a_m, μ_1, and μ_2 are constants. It follows from Eq. (18a) that all a_m for $m \geqq 3$ are zero, and hence, the only interacting modes are those involved in the internal resonance. There are two possibilities: either a_1 is zero and a_2 differs from zero, or both a_1 and a_2 differ from zero. In the present numerical example, which also has the 3:1 resonance, we show that multibranch solutions can exist for each case.

As a rule, not all the branches are stable. To investigate the stability, we superimpose an infinitesimal perturbation on the steady-state solution. From Eqs. (16) and (17), we obtain a set of linear equations with constant coefficients which govern the perturbations. If the real part of each eigenvalue of the coefficient matrix is negative, the solution is stable. When more than one stable solution exists, the initial conditions determine which solution gives the response. Jump phenomena are associated with the regions where more than one stable solution exists.

The final comments of this section are concerned with the ordering scheme. When the damping term is considered order one, regardless of the order of the forcing term, the first approximation agrees with the solution of the linearized problem. The complete solution, which can be obtained by a straightforward procedure, contains higher harmonics of the forcing frequency in the higher-order terms but none of the modal interactions mentioned above. Moreover, this solution does not exhibit any dependence of the frequency on the amplitude, as indicated by Eq. (18b) above. With this scheme, the influence of the damping term on the solution dominates the influence of the nonlinear terms. Consequently, this solution may bear little, or no, resemblance to the relatively large-amplitude, resonant response of a slightly damped system. It appears that the damping term in the present example should be order ε. (We note that when this is done for a linear problem, the approximate solution is a uniformly valid expansion of the exact solution.)

When the damping term is order ε, the forcing term must also be order ε when λ is near ω_m; otherwise, the first term may be unbounded. If λ is not near ω_m, the forcing term may be considered order one, and in this case,

$$\psi_{mo} = A_m(T_1)\exp(i\omega_m T_o) + P_m \exp(i\lambda T_o)/(\omega_m^2 - \lambda^2) + cc$$

The right-hand side of Eq. (13), which no longer contains the forcing term, now has, among others, factors of the form

$$\exp\{i[(\omega_n + \omega_p + \omega_q)T_o, (\omega_n - \omega_p + \omega_q)T_o, 3\lambda T_o, (\omega_n - \omega_p - \omega_q)T_o, (2\lambda - \omega_n), (\lambda - \omega_n - \omega_p)T_o, (\lambda - \omega_n + \omega_p)T_o]\}$$

Any combination of frequencies that equals, or nearly equals, ω_m can lead to modal interactions as well as to the so-called subharmonic, ultraharmonic, and subultraharmonic responses. For the harmonic resonances, the homogeneous solution is entrained by the particular solution and becomes part of the steady-state solution, in spite of damping. None of these possibilities is taken up here, but Bennett considered an ultraharmonic response.

Finally, we note that in similar situations involving cubic nonlinearities (Nayfeh et al.[21]) as well as quadratic nonlinearities (Nayfeh, Mook and Marshall,[23] and Mook, Marshall and Nayfeh[24]) the asymptotic results based on the present ordering scheme agree very closely with the results obtained by integrating the nonlinear equations numerically.

B. Application to Beams

Referring to Fig. 1, one can write the equations of motion as follows:

$$(\partial/\partial x)(n\cos\theta - q\sin\theta) = \rho A\, \partial^2 u/\partial t^2 \tag{20a}$$

$$(\partial/\partial x)(n\sin\theta + q\cos\theta) + F - C\,\partial w/\partial t = \rho A\, \partial^2 w/\partial t^2 \tag{20b}$$

and

$$\partial m/\partial x - n[(1+\partial u/\partial x)\sin\theta - \partial w/\partial x \cos\theta] - q[(1+\partial u/\partial x)\cos\theta + \partial w/\partial x \sin\theta] = -\rho I\, \partial^2\theta/\partial t^2 \tag{20c}$$

The constitutive relations are

$$m = -EI\,\partial\theta/\partial x \tag{20d}$$

and

$$n = EA\{[(1+\partial u/\partial x)^2 + (\partial w/\partial x)^2]^{1/2} - 1\} \tag{20e}$$

Shear deformations are neglected.

We introduce dimensionless variables (indicated by primes) as follows:

$$x = Lx', \qquad u = \varepsilon^2 L u' \tag{21a}$$

$$w = \varepsilon L w', \qquad t = Tt' \tag{21b}$$

The characteristic length L may be the actual length of the beam or the wavelength of a transverse oscillation. The perturbation parameter ε and the characteristic time T are defined as follows:

$$\varepsilon = r_o^2/L^2 \qquad \text{and} \qquad T^2 = \rho_o L^4/E_o r_o^2 \tag{21c}$$

where r_o, E_o, and ρ_o are the radius of gyration, the Young's modulus, and the density of the reference cross section, respectively.

Using Eqs. (21) and combining Eqs. (20a), (20b), and (20c), one can obtain, after dropping the primes,

$$\partial^2/\partial x^2(\alpha_1\,\partial^2 w/\partial x^2)+\alpha_2\,\partial^2 w/\partial t^2 = \varepsilon[\partial/\partial x(\alpha_2 G\,\partial w/\partial x)+P-2c\,\partial w/\partial t] \quad (22a)$$

where

$$G=\frac{k\int_o^l\left(\frac{\partial w}{\partial x}\right)^2 dx}{2\left(k\int_o^l\frac{dx}{\alpha_2}+\frac{1}{\alpha_2(l)}\right)}=B\int_o^l\left(\frac{\partial w}{\partial x}\right)^2 dx \quad (22b)$$

with k the "spring constant" so that

$$ku(l,t)+[(1+\partial u/\partial x)^2+(\partial w/\partial x)^2]^{1/2}-1=0 \quad (22c)$$

$$\alpha_1 = I/I_o \quad (22d)$$

$$\alpha_2 = A/A_o \quad (22e)$$

$$P = FL^7/r_o{}^6 E_o A_o \quad (22f)$$

and

$$c = CL^4/r_o{}^3 A(\rho_o E_o)^{1/2} \quad (22g)$$

We note that to this order of approximation the linear terms which account for shear deformation and rotary inertia can be combined. This term is formally the same order as the nonlinear term which accounts for stretching. However, the linear term can only affect the solution slightly (see, e.g., Timoshenko[25] and Nayfeh et al.[21]), while the nonlinear term can alter the character of the solution completely.

The deflection must also satisfy boundary conditions of the form

$$B_o(w)=0 \quad \text{at} \quad x=0 \quad \text{and} \quad B_l(w)=0 \quad \text{at} \quad x=l$$

Eq. (22a) and these boundary conditions are the forms of Eqs. (1) and (2) which apply to beams.

The linear, free-oscillation modes in the expansion given in Eq. (3) are solutions of

$$(d^2/dx^2)(\alpha_1 d^2\phi_m/dx^2)-\omega_m{}^2\alpha_2\phi_m=0 \quad (23a)$$

where

$$B_o(\phi_m)=0 \quad \text{at} \quad x=0 \quad \text{and} \quad B_l(\phi_m)=0 \quad \text{at} \quad x=l \quad (23b)$$

Following the general procedure, one finds that the coefficients in Eq. (5) are given by

$$\alpha_{mnpq}=\left(\int_o^l \phi_m\frac{d^2\phi_n}{dx^2}dx\right)\left(\int_o^l\frac{d\phi_p}{dx}\frac{d\phi_q}{dx}dx\right)(B) \quad (24)$$

Next we discuss how these coefficients are obtained.

C. Finite-Element Solution of the Linear Problem

The finite-element selected coincides with the Euler-Bernoulli beam theory employed in the perturbation analysis. Thus, using the same nondimensional terms as above, one finds the following stiffness matrix (see, e.g., Przemieniecki[26])

$$k=\left(\frac{\gamma}{\alpha}\right)^3\begin{bmatrix}12 & & & \text{sym}\\ 6\alpha & 4\alpha^2 & & \\ -12 & -6\alpha & 12 & \\ 6\alpha & 2\alpha^2 & -6\alpha & 4\alpha^2\end{bmatrix}$$

Here, γ is the ratio of the element height to the characteristic height, and α is the ratio of the element length to the characteristic length. To take full advantage of the bandedness of the assembled stiffness matrix, we use the diagonal form of the associated element mass matrix, shown below

$$m=\frac{\alpha\gamma}{2}\begin{bmatrix}1 & & & \\ & \alpha^2/12 & & \\ & & 1 & \\ & & & \alpha^2/12\end{bmatrix}$$

The discretization of the beam and the apportionment of the element mass to the surrounding nodes is done so that the node point is always located at the center of mass of the mass it represents. In this manner the translational kinetic energy is accurately taken into account.

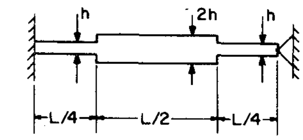

Fig. 2 Beam used in the numerical example.

The resulting linear eigenvalue problem is as follows:

$$(K-\lambda M)V=0$$

where K and M are the assembled stiffness and mass matrices, λ is the eigenvalue, and V is the associated mode shape. Making the transformation $\tilde{V}=M^{1/2}V$ yields the following canonical form:

$$(\tilde{K}-\lambda I)\tilde{V}=0$$

where $\tilde{K}=M^{-1/2}KM^{-1/2}$. Because M is diagonal, the bandwith of $\tilde{K}$ is equal to the bandwidth of K. Algorithms exist whereby all the eigenvalues and selected eigenvectors of this system can be computed, requiring core storage for the banded $\tilde{K}$ matrix and the computed eigenvectors only. Thus, this method can be extended to systems having more degrees of freedom on an in-core basis.

For the present case all the eigenvectors and eigenvalues are found by first reducing $\tilde{K}$ to tridiagonal form and then applying the implicit QL method (see, e.g., Martin and Wilkinson[27]). Comparisons with exact solutions show that agreement to three significant figures is not difficult to achieve. (For the present problem, the discretization was refined until convergence to three significant figures occurred.) The integrals in Eq. (24) are evaluated exactly using the element shape function.

III. Numerical Example

As an example, we consider the beam shown in Fig. 2. Using the finite-element method described previously, we find

$$\omega_1 = 18.90, \quad \omega_2 = 58.22, \quad \text{and} \quad \varepsilon\sigma_1 = 0.0268$$

Taking $B=\frac{2}{3}$ (which corresponds to immovable ends), normalizing the eigenfunctions so that

$$\int_o^l \alpha_2\phi_n{}^2\,dx = 1$$

and using Eq. (24), we find that the coefficients in Eqs. (16) and (17) are as follows:

$$\delta_1=-77.53, \quad \delta_2=-25.80, \quad \gamma_{11}=-101.7$$

$$\gamma_{12}=\gamma_{21}=-372.7, \quad \text{and} \quad \gamma_{22}=1909$$

The resulting, nonlinear, transcendental equations are solved with a Newton-Raphson procedure for two cases—λ near ω_1 and λ near ω_2.

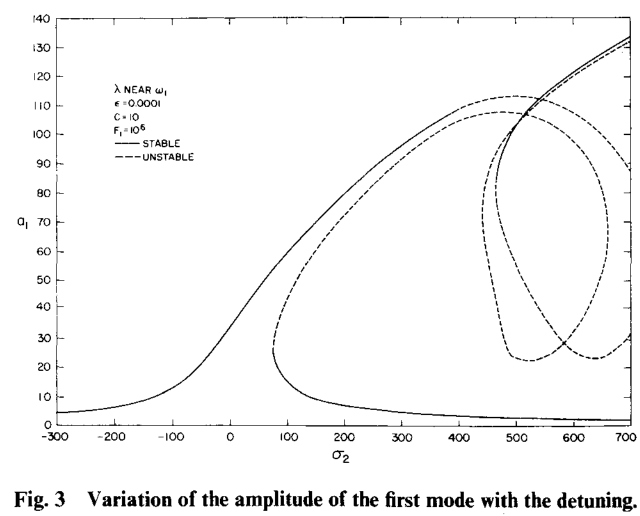

Fig. 3 Variation of the amplitude of the first mode with the detuning.

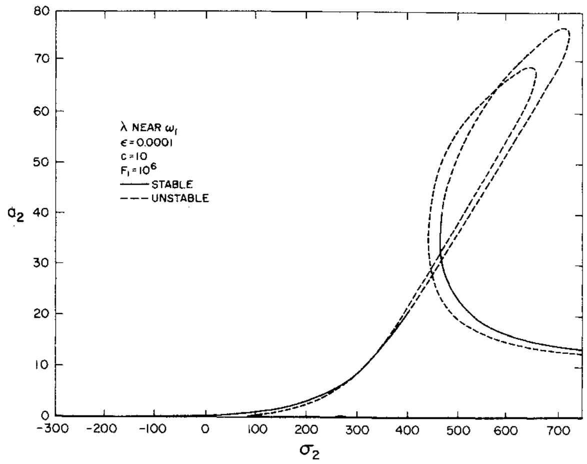

Fig. 4 Variation of the amplitude of the second mode with the detuning.

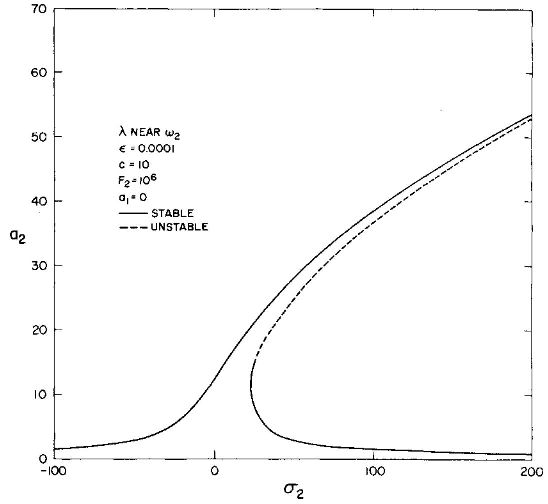

Fig. 7 Variation of the amplitude of the second mode with the detuning, when the first mode is not excited.

A. λ Near ω_1

In this case, the forcing terms are deleted from Eqs. (17) and added to Eqs. (16), and the definitions of σ_2 and μ_2 are changed as follows:

$$\lambda = \omega_1(1+\varepsilon\sigma_2) \quad \text{and} \quad \mu_2 = \omega_1\sigma_2 T_1 - \beta_1$$

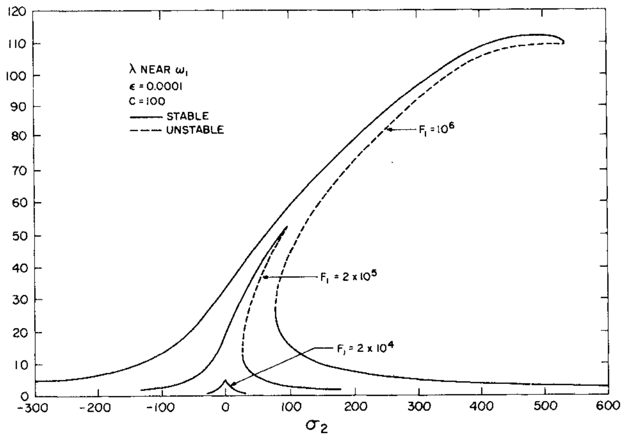

Fig. 5 Variation of the amplitude of the first mode with the detuning for various amplitudes of the forcing function.

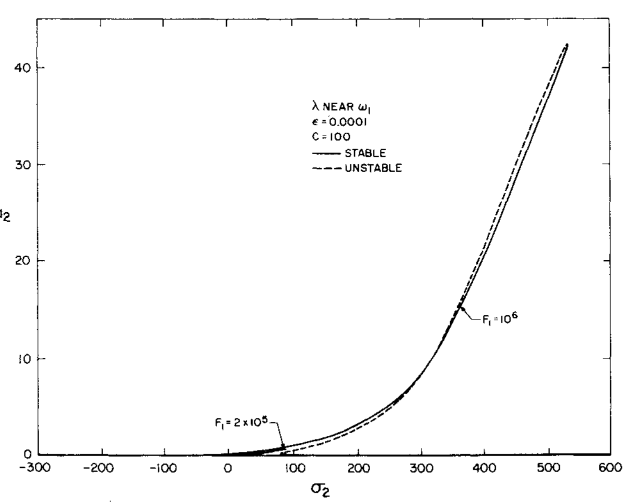

Fig. 6 Variation of the amplitude of the second mode with the detuning for various amplitudes of the forcing function.

In Figs. 3 and 4, we show a_1 and a_2 as functions of the detuning parameter σ_2. We note the complex structure of the response at large values of σ_2, which is due to the internal resonance and could not be predicted by a single-mode expansion. The loop appearing at large values of σ_2 disappears when the damping is increased and/or the amplitude of the excitation is decreased. This is illustrated in Figs. 5 and 6. For the largest curves, the amplitude of the excitation is the same as in Figs. 3 and 4, but the damping is tenfold larger.

The effect of decreasing the amplitude of the excitation is also illustrated. Though a_2 cannot be zero, it decreases much more rapidly than a_1 when the amplitude of the excitation decreases. The values of a_2 which correspond to the smallest curve in Fig. 5 are too small to be plotted in Fig. 6. Figures 5 and 6 also illustrate the fact that the present solution approaches the solution of the linearized problem as the amplitude of the excitation decreases.

B. λ Near ω_2

In this case, Eqs. (16) and (17), apply as they appear previously. There are two possibilities: either a_1 is zero and a_2 is different from zero, or both a_1 and a_2 are different from zero.

When a_1 is zero, a_2 varies with the detuning σ_2 as shown in Fig. 7. We note that if a_1 is zero, a_2 is the solution of

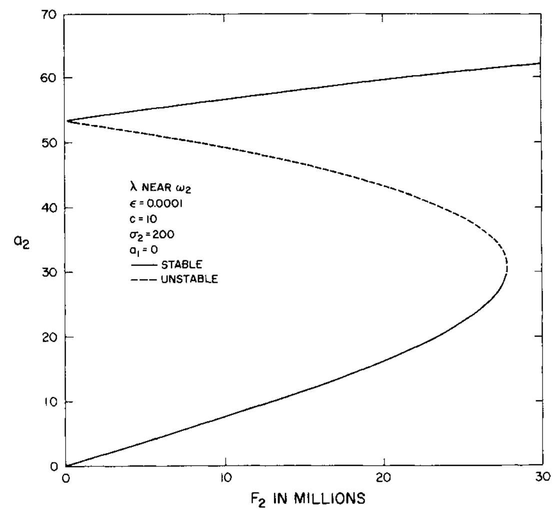

Fig. 8 Variation of the amplitude of the second mode with amplitude of the excitation, when the first mode is not excited.

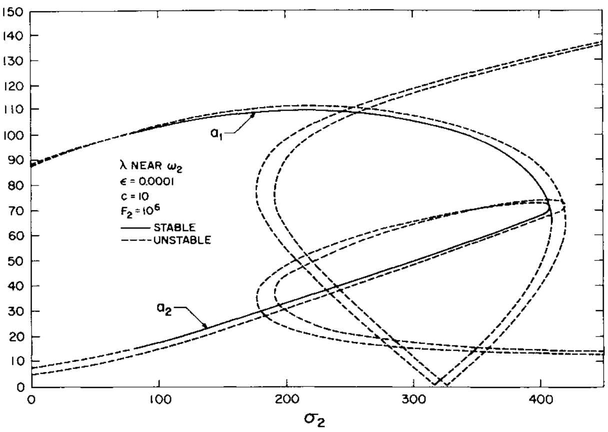

Fig. 9 Variation of the amplitudes of the first and second modes with the detuning.

Duffing's equation with a damping and a forcing term. The variation of a_2 with the amplitude of the excitation is shown in Fig. 8. The upper branch does not quite reach back to the axis. Consequently, when the force decreases, the response approaches the solution of the linearized problem.

When both a_1 and a_2 are different from zero, the corresponding plots are shown in Figs. 9 and 10. We note that if a_1 is not zero, it can be five to ten times larger than a_2. In this case, the response is similar to that produced when the frequency of the excitation is near the fundamental frequency. This is illustrated in Fig. 11 where we show how the two possible deflection curves vary with time (The deflections are not plotted to the same scale as the distance along the beam.) Finally, we note that the excitation amplitude must exceed a certain threshold value before a_1 can be different from zero. The threshold value increases as the damping increases; in this example, the damping is very small.

IV. Conclusions

A numerical-perturbation technique has been proposed for the determination of the nonlinear forced response of general structural elements. The present technique extends that of Nayfeh et al.[21] to structural elements having complicated boundaries and/or composition.

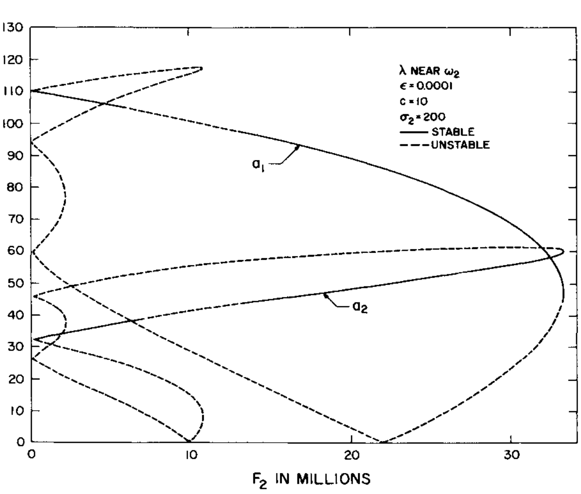

Fig. 10 Variation of the amplitudes of the first and second modes with the amplitude of the excitation.

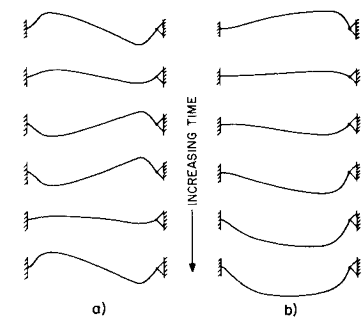

Fig. 11 The variation of the deflection with time when λ is near ω_2 ($\sigma_2 = 120$, $\varepsilon = 0.0001$, $a_1 = 105$, $a_2 = 20.5$); a) $w = a_2\phi_2 \cos(\lambda t - \mu_2)$ ($a_1 = 0$), and b) $w = a_1\phi_1 \cos[(\lambda t - \mu_1 - \mu_2)/3] + a_2\phi_2 \cos(\lambda t - \mu_2)$.

According to this technique, the problem is represented as a nonlinear temporal problem and a linear spatial problem. The spatial problem can be solved by using any one of a number of existing finite-difference or finite-element techniques, while the temporal problem is solved by using the method of multiple scales. The technique was demonstrated in this paper by applying it to the analysis of the large-amplitude forced vibrations of a hinged-clamped beam having a discontinuous cross section.

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Prediction of Unsteady Airloads for Oblique Blade-Gust Interaction in Compressible Flow

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The techniques of Galilean-Lorentz transformation and matched asymptotic expansions are used to simplify the procedure of calculating the lift and pressure distribution induced on an infinite-span thin wing interacting with an oblique sinusoidal gust in subsonic flow. This technique requires that the product of the flow Mach number and the reduced frequency is small. Under this condition, the inner region of the transformed space behaves as an incompressible flow so that existing incompressible flow theories can be used as a basis to construct closed-form solutions for the airload induced on the wing. This approach is an extension of the GASP approximation developed by Sears and Amiet. Results are obtained for both the magnitude and the phase of the unsteady lift due to interaction with gust. These results are compared with available numerical results. Some discrepancies are noted and discussed.

Nomenclature

a	= speed of sound
b_o	= wing semichord
b	= wing semichord in the Galilean-Lorentz (G-L) transformed space
$C(S)$	= Theodorsen function $[=H_1^{(2)}(S)/\{H_1^{(2)}(S)+iH_0^{(2)}(S)\}]$
C_L	= lift coefficient
C_p	= pressure coefficient
g	$= \{1-(\tan\Lambda^T/M)^2\}^{1/2}$
g'	$= -ig$
$H_n^{(1,2)}(x)$	= nth order Hankel function of the first or the second kind
$I_n(x)$, $K_n(x)$	= nth order modified Bessel functions of the first and the second kind
$J_n(x)$	= nth order Bessel function of the first kind
k	= gust wave number
k_o	= normalized gust wave number $(=kb_o)$
k_x, k_y	= chordwise and spanwise components of the gust wave number
k_2	$= k_o \sin\Lambda$
K, K_o, K_X, K_Y, K_2	= counterparts of k, k_o, k_x, k_y, k_2 in the G-L transformed space
K'	$= K_2 + i\sigma$
M	= freestream Mach number
M_c	= airload convection Mach number $(=M/\sin\Lambda)$
P	= amplitude of the pressure coefficient in the G-L transformed space
r_o, θ	= cylindrical coordinates in the outer region
$\bar{r}$	= cylindrical coordinate in the inner region $(\bar{r} = r_o/\varepsilon)$
s	= reduced frequency $(=\omega b_o/U)$
S	= reduced frequency in the G-L transformed space $(=\Omega b/U = s/\beta^2)$
T_A	= aerodynamic transfer function $(=C_L/\{2\pi\alpha_o/\beta\})$
T_o	= aerodynamic transfer function for the incompressible flow
$T_{o_{2D}}$	= aerodynamic transfer function for the two-dimensional incompressible flow
U	= freestream velocity
w	= gust upwash velocity
x, y, z, t	= coordinates of the original space attached to the wing
x', y', z', t'	= coordinates of the Galilean transformed space convected with the freestream
X, Y, Z, T	= coordinates of the G-L transformed space attached to the wing
$\bar{X}, \bar{Y}, \bar{Z}, \bar{T}$	= normalized inner variables in the G-L transformed space
X_o, Y_o, Z_o, T_o	= normalized outer variables in the G-L transformed space
α	= incident angle
α_o	= incident angle at the midchord

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